



# Quasinormal modes and second order thermodynamic phase transition for Reissner–Nordström black hole

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## Abstract

The relation between the quasinormal modes (QNMs) and the second order thermodynamic phase transition (SOTPT) for the Reissner–Nordström (RN) black hole is studied. It is shown that the quasinormal frequencies of the RN black hole start to get a spiral-like shape in the complex  $\omega$  plane and both the real and imaginary parts become the oscillatory functions of the charge if the real part of the quasinormal frequencies arrives at its maximum at the second order phase transition point of Davies for given overtone number and angular quantum number. That is to say, we can find out the SOTPT point from the QNMs of the RN black hole. The fact shows that the quasinormal frequencies carry the thermodynamical information of the RN black hole.

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The QNMs of a black hole are defined as proper solutions of the perturbation equations belonging to certain complex characteristic frequencies which satisfy the boundary conditions appropriate for purely ingoing waves at the event horizon and purely outgoing waves at infinity [1]. They are entirely fixed by the structure of the background spacetime and irrelevant of the initial perturbations [1,2]. Thus, it is generally believed that QNMs carry the footprint to directly identify the existence of a black hole. Meanwhile, the study of QNMs may lead to a deeper understanding of the thermodynamical properties of black holes in loop quantum gravity [3,4], as well as the QNMs of anti-de Sitter black holes have a direct interpretation in terms of the dual conformal field theory [5–7].

On the other hand, one of important characteristics of a black hole is its thermodynamical properties. It is well known that the heat capacity of the Schwarzschild black hole is always negative and so the black hole is thermodynamically unstable. But

for the RN black hole, the heat capacity is negative in some parameter region and positive in other region. Davies pointed out that the phase transition appears in black hole thermodynamics and the SOTPT takes place at the point where the heat capacity diverges [8–10].

Because the QNMs of a black hole are entirely fixed by the structure of the background spacetime and the SOTPT is only related to the parameters of the black hole, an interesting question is whether there are some relations between them. The aim of this Letter is to study this question for the RN black hole, and we find that the quasinormal frequencies do carry the thermodynamical information of the black hole.

Assuming that the azimuthal and time dependence of the fields will be the form  $e^{-i(\omega t - m\varphi)}$  and using the Newman–Penrose formalism [11], we can obtain the separated equations for massless scalar, Dirac, and Rarita–Schwinger (RS) perturbations around a RN black hole [12–14]

$$\begin{aligned} [\Delta \mathcal{D}_{1-s} \mathcal{D}_0^\dagger + 2(2s-1)i\omega r - (\lambda + 2s)] \Delta^s R_s &= 0, \\ [\mathcal{L}_{1-s}^\dagger \mathcal{L}_s + (\lambda + 2s)] S_s &= 0 \quad (s = 0, 1/2, 3/2) \end{aligned} \quad (1)$$

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$$[\Delta \mathcal{D}_{1+s}^\dagger \mathcal{D}_0 + 2(2s+1)i\omega r - \lambda]R_s = 0, \\ (\mathcal{L}_{1+s}\mathcal{L}_{-s}^\dagger + \lambda)S_s = 0 \quad (s=0, -1/2, -3/2) \quad (2)$$

where  $\Delta = r^2 - 2Mr + Q^2$ ,  $M$  and  $Q$  represent the mass and charge of the black hole, and  $\lambda$  is the angular separation constant [15–17],

$$\lambda = \begin{cases} (l-s)(l+s+1), & l=|s|, |s|+1, \dots, \\ (j-s)(j+s+1), & j=|s|, |s|+1, \dots, \end{cases} \quad (3)$$

where  $l$  and  $j$  are the quantum numbers characterizing the angular distribution for the boson and fermion perturbations, respectively. Introducing a usual tortoise coordinate  $dr_* = r^2/\Delta dr$  and resolving the equation in the form  $R_s = \Delta^{-s/2} \times \Psi_s/r$ , we can rewrite the radial wave equations in Eqs. (1) and (2) as

$$\frac{d^2 \Psi_s}{dr_*^2} + [\omega^2 - V]\Psi_s = 0, \quad (4)$$

with

$$V = is\omega r^2 \frac{d}{dr} \frac{\Delta}{r^4} + \frac{(s+\lambda)\Delta + (\frac{s}{2} \frac{d\Delta}{dr})^2}{r^4} + \frac{\Delta}{r^3} \frac{d}{dr} \frac{\Delta}{r^2}. \quad (5)$$

The boundary conditions on wave function  $\Psi_s$  at the horizon and infinity can be expressed as

$$\Psi_s \sim \begin{cases} (r-r_+)^{-\frac{s}{2} - \frac{i\omega}{2\kappa_+}}, & r \rightarrow r_+, \\ r^{-s+i\omega} e^{i\omega r}, & r \rightarrow +\infty, \end{cases} \quad (6)$$

where  $\kappa_{\pm} = (r_+ - r_-)/(2r_{\pm}^2)$  is the surface gravity on the horizons  $r_{\pm}$ . A solution to Eq. (4) that has the desired behavior at the boundary can be written as

$$\Psi_s = r(r-r_+)^{-\frac{s}{2} - \frac{i\omega}{2\kappa_+}} (r-r_-)^{-1-\frac{s}{2}+2i\omega+\frac{i\omega}{2\kappa_-}} \\ \times e^{i\omega(r-r_-)} \sum_{m=0}^{\infty} a_m \left( \frac{r-r_+}{r-r_-} \right)^m. \quad (7)$$

If we take  $r_+ + r_- = 1$ , the sequence of the expansion coefficients  $\{a_m; m=1, 2, \dots\}$  is determined by a three-term recur-

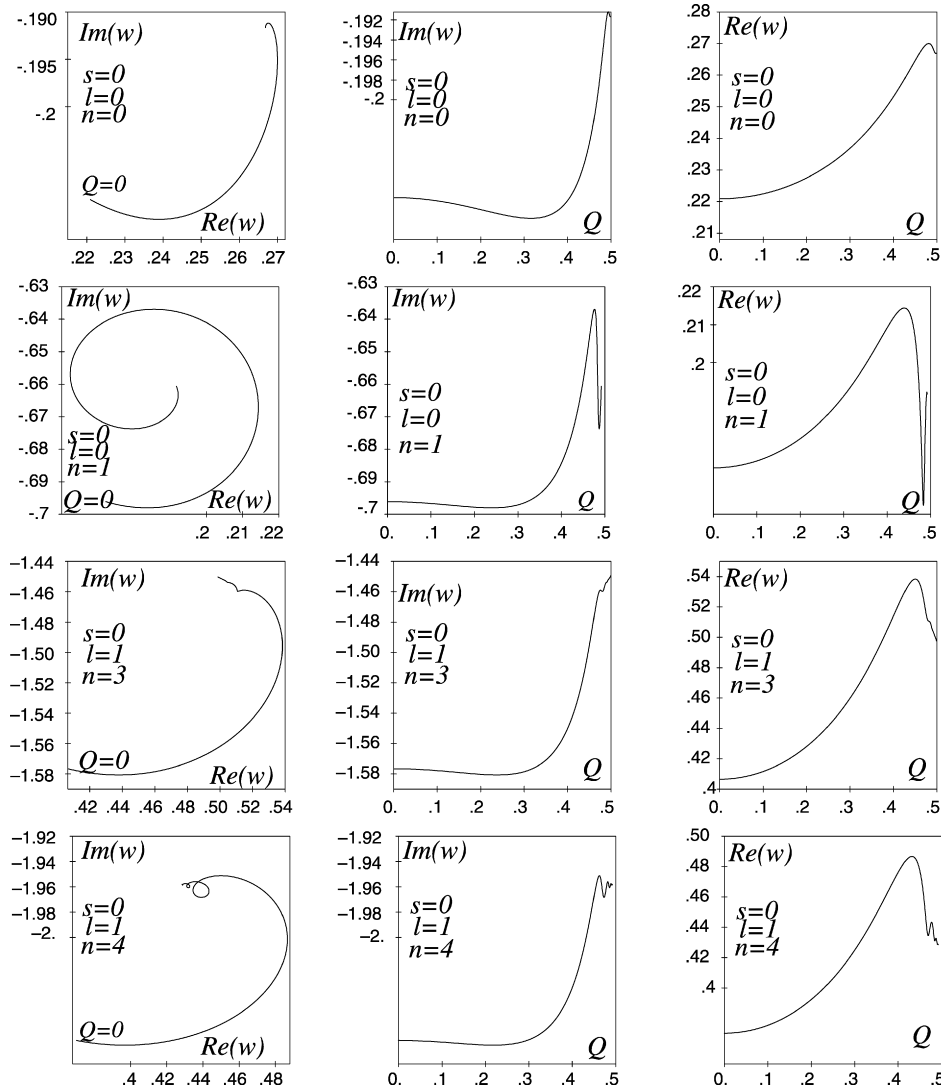


Fig. 1. Left four panels show trajectories in the complex  $\omega$  plane of the scalar quasinormal frequencies of the RN black hole for  $l=0, n=1$  and  $l=1, n=3, 4$ . The other panels draw the imaginary part  $\text{Im}(\omega)$  and real part  $\text{Re}(\omega)$  of the quasinormal frequencies versus the charge  $Q$ .

rence relation starting with  $a_0 = 1$ :

$$\begin{aligned}\alpha_0 a_1 + \beta_0 a_0 &= 0, \\ \alpha_m a_{m+1} + \beta_m a_m + \gamma_m a_{m-1} &= 0, \quad m = 1, 2, \dots\end{aligned}\quad (8)$$

The recurrence coefficients  $\alpha_m$ ,  $\beta_m$  and  $\gamma_m$  are given in terms of  $m$  and the black hole parameters by

$$\begin{aligned}\alpha_m &= m^2 + (C_0 + 1)m + C_0, \\ \beta_m &= -2m^2 + (C_1 + 2)m + C_3, \\ \gamma_m &= m^2 + (C_2 - 3)m + C_4 - C_2 + 2\end{aligned}\quad (9)$$

and the intermediate constants  $C_m$  are defined by

$$\begin{aligned}C_0 &= 1 - s - i\omega - i\omega B, \\ C_1 &= -4 + 2i\omega(2 + b) + 2i\omega B, \\ C_2 &= s + 3 - 3i\omega - i\omega B, \\ C_3 &= \omega^2(4 + 2b - 4r_+ r_-) - s - 1 + (2 + b)i\omega \\ &\quad - \lambda^2 + (2\omega + i)\omega B,\end{aligned}$$

$$C_4 = s + 1 + 2\left(i\omega - s - \frac{3}{2}\right)i\omega - (2\omega + i)\omega B, \quad (10)$$

where  $B = (r_+^2 + r_-^2)/(r_+ - r_-)$ . The series in (7) converges and the  $r \rightarrow +\infty$  boundary condition (6) is satisfied if, for a given  $s$  and  $\lambda$ , the frequency  $\omega$  is a root of the continued fraction equation

$$\begin{aligned}\left[\beta_m - \frac{\alpha_{m-1}\gamma_m}{\beta_{m-1}-} \frac{\alpha_{m-2}\gamma_{m-1}}{\beta_{m-2}-} \dots \frac{\alpha_0\gamma_1}{\beta_0}\right] \\ = \left[\frac{\alpha_m\gamma_{m+1}}{\beta_{m+1}-} \frac{\alpha_{m+1}\gamma_{m+2}}{\beta_{m+2}-} \frac{\alpha_{m+2}\gamma_{m+3}}{\beta_{m+3}-} \dots\right], \quad (m = 1, 2, \dots).\end{aligned}\quad (11)$$

This leads to a simple method to find quasinormal frequencies of the RN black hole—defining a function which returns the value of the continued fraction for an initial guess at the frequency, and then use a root finding routine to find the zeros of this function in the complex  $\omega$  plane. The frequency for which happens is a quasinormal frequency [18,19].

Figs. 1–3 describe the QNMs for the scalar ( $s = 0$ ), Dirac ( $s = -1/2$ ), and RS fields ( $s = -3/2$ ) obtained by the con-

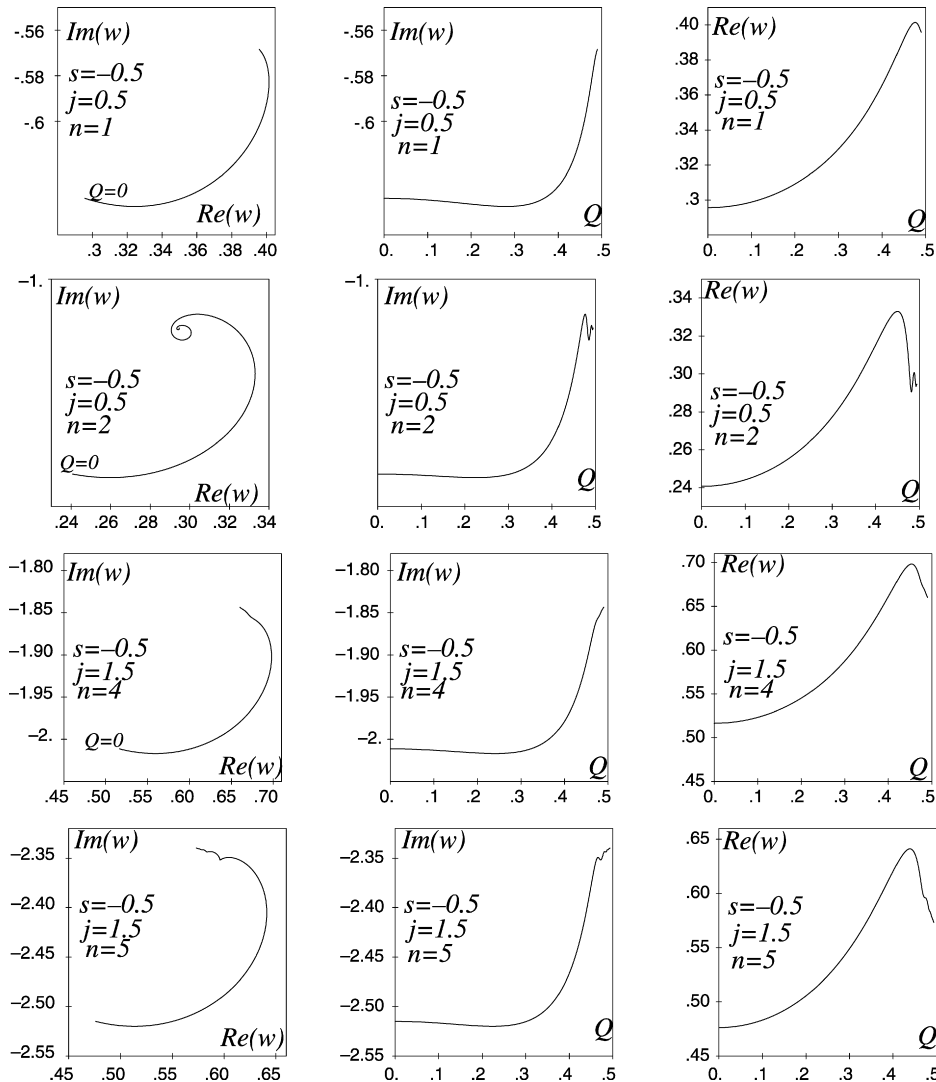


Fig. 2. Left four panels show trajectories in the complex  $\omega$  plane of the Dirac quasinormal frequencies for  $j = 1/2, n = 1, 2$  and  $j = 3/2, n = 4, 5$ . The other panels draw the imaginary part  $\text{Im}(\omega)$  and real part  $\text{Re}(\omega)$  of the quasinormal frequencies versus the charge  $Q$ .

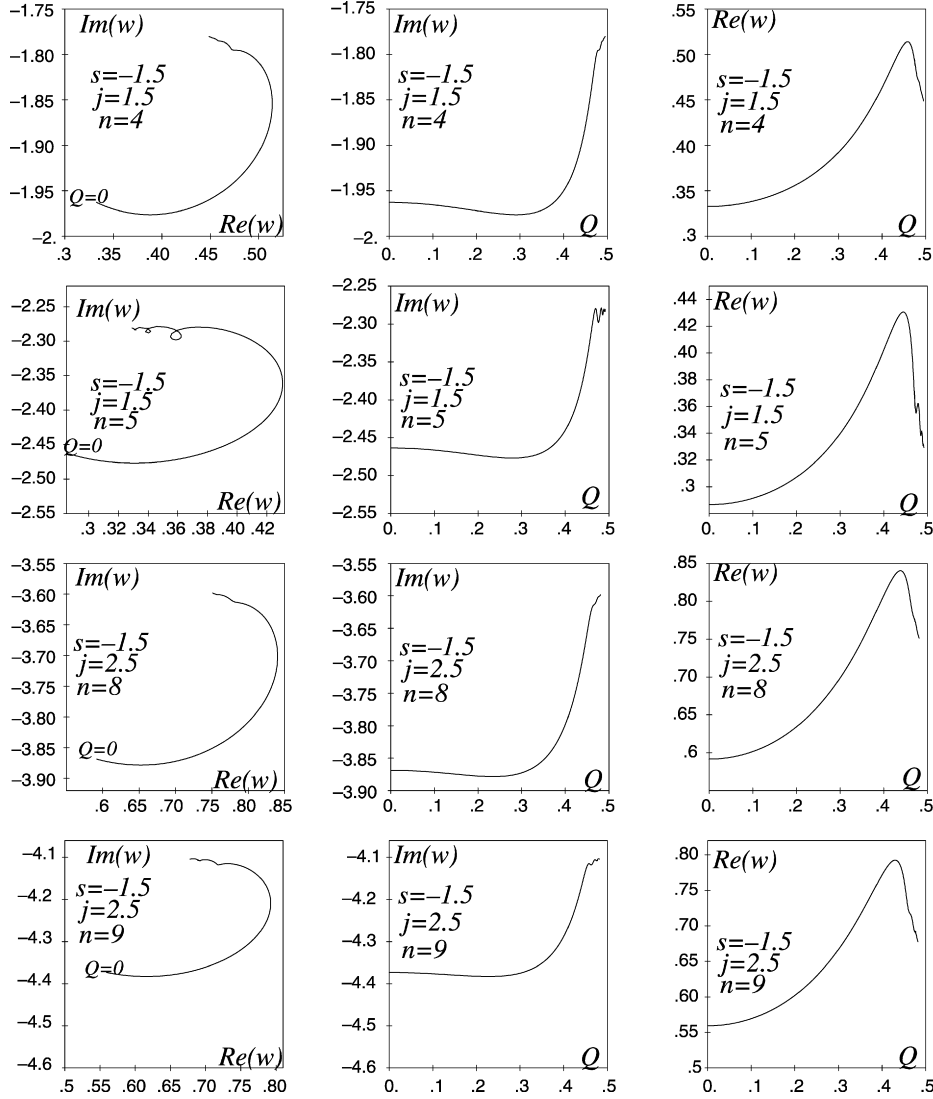


Fig. 3. Left four panels show trajectories in the complex  $\omega$  plane of the RS quasinormal frequencies for  $j = 3/2, n = 4, 5$  and  $j = 5/2, n = 8, 9$ . The other panels draw the imaginary part  $\text{Im}(\omega)$  and real part  $\text{Re}(\omega)$  of the quasinormal frequencies versus the charge  $Q$ .

tinued fraction method respectively. In Fig. 1 (or Figs. 2, 3) left four panels show trajectories in the complex  $\omega$  plane of the scalar (or Dirac, RS) quasinormal frequencies of the RN black hole with different angular quantum number and overtone number. The other panels draw the imaginary part  $\text{Im}(\omega)$  and real part  $\text{Re}(\omega)$  of the quasinormal frequencies versus the charge  $Q$ . These figures tell us that in the complex  $\omega$  plane the quasinormal frequencies will get a spiral-like shape as the charge  $Q$  increases to its extremal value when the overtone number equals to or exceeds a critical value  $n_c$  for a fixed angular quantum number (say,  $n_c = 1$  with  $l = 0$  and  $n_c = 4$  with  $l = 1$  for the scalar field,  $n_c = 2$  with  $j = 1/2$  and  $n_c = 5$  with  $j = 3/2$  for the Dirac field, and  $n_c = 5$  with  $j = 3/2$  and  $n_c = 9$  with  $j = 5/2$  for the RS field.), and at same time both the real and imaginary parts become the oscillatory functions of the charge. The critical value of the overtone number  $n_c$  increases as the angular quantum number  $l$  (or  $j$ ) increases for a given perturbation, and it also increases as the absolute value of spins  $|s|$  increases for the same level of the angular quantum number

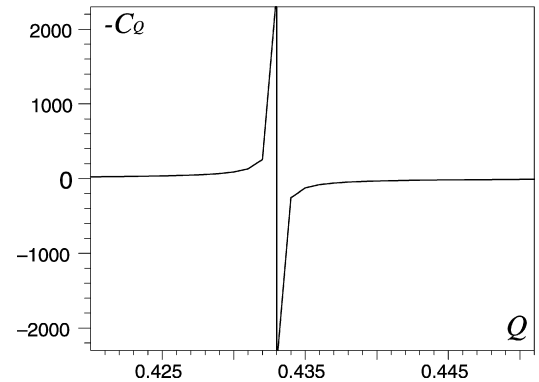


Fig. 4. The panel shows trajectory of  $-C_Q$  versus  $Q$  near the singular point of the heat capacity  $Q_{\text{sp}} = 0.433$ .

(here the least level is defined as  $l = 0, j = 1/2$ , and  $j = 3/2$  for the scalar, Dirac, and RS fields, respectively).

On the other hand, we know that the heat capacity  $C_Q$  of the RN black hole is given by [8,9]

$$C_Q = T \left( \frac{\partial S}{\partial T} \right)_Q = \frac{TS^3 M}{\pi Q^4/4 - S^3 T^2}, \quad (12)$$

where  $T$  is the temperature and  $S$  is the entropy of the black hole. It is obvious that the singular point of the heat capacity (SPHC) occurs at  $\pi Q^4/4 - S^3 T^2 = 0$ , i.e.,  $Q_{\text{sp}} = \sqrt{3}M/2 \approx 0.4330127$ . Fig. 4 shows that the heat capacity is negative in the region  $Q < Q_{\text{sp}}$  and positive in the region  $Q > Q_{\text{sp}}$ . Davies [8–10] showed that the SPHC is just the SOTPT point of the black hole thermodynamics.

To find the relation between the QNMs and the SOTPT point, we present critical point  $Q_{\text{cp}}$ , which is the value of the charge at which the real part of the first oscillatory quasinormal frequencies arrives at its maximum, with the different critical overtone number  $n_c$  for the scalar, Dirac and RS fields

Table 1  
The values of the charge for  $Q_{\text{cp}}$

	Scalar field ( $s = 0$ )	
$(l, n_c)$	(0, 1)	(1, 4)
$Q_{\text{cp}}$	0.438	0.432
$\frac{ \Delta Q }{Q_{\text{cp}}}$	1.15%	0.23%
	Dirac field ( $s = -1/2$ )	
$(j, n_c)$	(1/2, 2)	(3/2, 5)
$Q_{\text{cp}}$	0.442	0.436
$\frac{ \Delta Q }{Q_{\text{cp}}}$	2.08%	0.69%
	Rarita–Schwinger field ( $s = -3/2$ )	
$(j, n_c)$	(3/2, 5)	(5/2, 9)
$Q_{\text{cp}}$	0.444	0.430
$\frac{ \Delta Q }{Q_{\text{cp}}}$	2.55%	0.69%

in Table 1. From the table we find that the SPHC  $Q_{\text{sp}}$  is in good agreement with the critical point  $Q_{\text{cp}}$  because the difference between  $Q_{\text{sp}}$  and  $Q_{\text{cp}}$  is less than 2.55%. Besides, for the Schwarzschild black hole there is no SPHC because its heat capacity is always negative and there is no critical point because its QNMs never get a spiral-like.

To compare them farther more, we take  $K = \frac{d\omega_I}{d\omega_R}$  as the slope of the QNMs. For  $Q < Q_{\text{cp}}$ , Fig. 5 shows that the QNMs have a positive slope and hence large values of  $\omega_R$  correspond to large values of  $\omega_I$ . However, for  $Q > Q_{\text{cp}}$ , the slope is negative and hence large values of  $\omega_R$  correspond to small values of  $\omega_I$ . At  $Q = Q_{\text{cp}}$ , the slope becomes infinity. The transition from negative to positive value of  $K$  occurs via a infinite discontinuity, characteristic of a second order phase transition. The trajectories of the slope of the QNMs take the same form as that of  $-C_Q$  versus  $Q$ .

We all know that there are two characteristic parameters for any perturbation of a black hole background: the oscillation time scale  $\tau_R = 1/\omega_R$  and the damping time scale  $\tau_I = 1/|\omega_I|$ . Near the critical point, although the increment of  $\tau_I$  increases monotonously as  $Q$  increases, the increment of  $\tau_R$  decreases as  $Q$  increases for  $Q < Q_{\text{cp}}$  and it does not change at all at  $Q = Q_{\text{cp}}$ , but it increases as  $Q$  increases for  $Q > Q_{\text{cp}}$ . It is curious that the change of the increment of  $\tau_R$  presented here is similar to that of the temperature of the black hole when it takes the same quality of heat. If we take a slope of the time scale as  $K_\tau = d\tau_I/d\tau_R$ , the trajectory of the slope takes the same form as that of  $C_Q$  versus  $Q$  because  $K_\tau = -K(\omega_R/\omega_I)^2$ . Above discussions show us that the critical point of the QNMs may be associated with the SOTPT point.

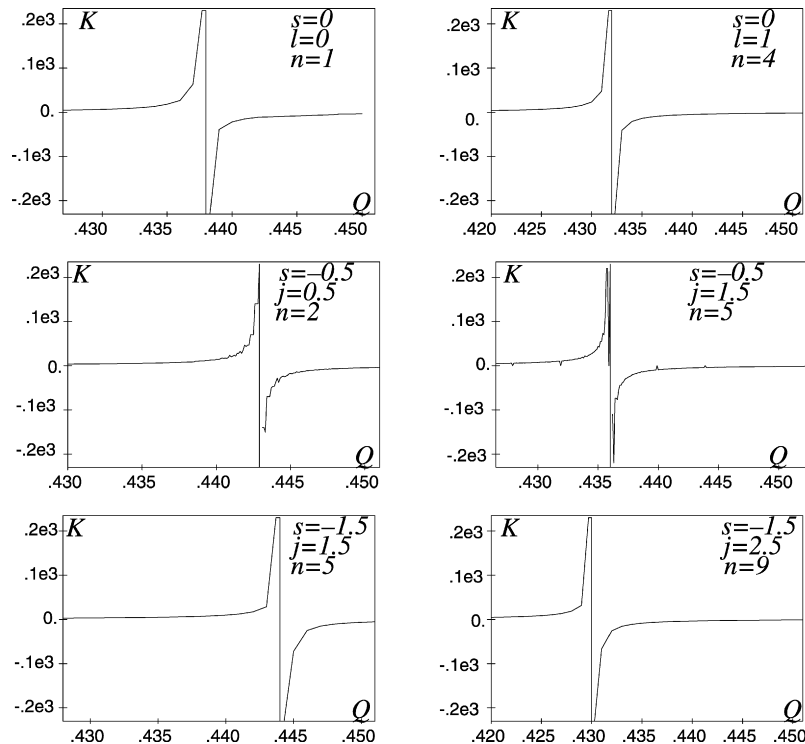


Fig. 5. The panels show trajectories of the slope  $K = d\omega_I/d\omega_R$  versus  $Q$  near the critical point  $Q_{\text{cp}}$  for the first oscillatory QNMs of the scalar, Dirac, and RS fields.

In summary, we study the relation between the QNMs and the SOTPT for the RN black hole and find the following results: if the real part of the quasinormal frequencies arrives at its maximum at the SOTPT point for given overtone number and angular quantum number, the QNMs will start to get a spiral-like shape in the complex  $\omega$  plane, and both the real and imaginary parts will become the oscillatory functions of the charge. The QNMs will (not) take a spiral-like shape if the angular quantum number or the overtone number larger (less) than this given value. If a black hole does not possess the SOTPT point, its QNMs never take a spiral-like shape in the complex  $\omega$  plane. Besides the fact that the critical point  $Q_{cp}$  of the QNMs is in good agreement with the SOTPT point  $Q_{sp}$ , the transition from negative to positive value of the slope  $K$  ( $K_\tau$ ) of the QNMs (time scale) occurs via a infinite discontinuity, and the trajectories of  $K$  and  $K_\tau$  take the same form as that of the heat capacity. These facts show that the critical point of the QNMs may be associated with the SOTPT point and the quasinormal frequencies carry the thermodynamical information of the RN black hole.

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